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A non-commuting twist in the partition function

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Abstract

We compute a twisted index for an orbifold theory when the twist generating group does not commute with the orbifold group. The twisted index requires the theory to be defined on moduli spaces that are compatible with the twist. This is carried for CHL models at special points in the moduli space where they admit dihedral symmetries. The commutator subgroup of the dihedral groups are cyclic groups that are used to construct the CHL orbifolds. The residual reflection symmetry is chosen to act as a “twist” on the partition function. The reflection symmetries do not commute with the orbifolding group and hence we refer to this as a non-commuting twist. We count the degeneracy of half-BPS states using the twisted partition function and find that the contribution comes mainly from the untwisted sector. We show that the generating function for these twisted BPS states are related to the Mathieu group M_{24} .

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1 Introduction

The microscopic counting of black hole entropy in four-dimensional string theories with $\mathcal{N} = 4$ supersymmetry has turned out to have a surprisingly rich structure [1, 2]. This has provided connections to modular forms, Lie algebras [3, 4] as well as sporadic groups [5, 6]. Due to the large amount of supersymmetry, these theories work as “laboratories” for us to test ideas that presumably should continue to work in situations with fewer super symmetries. This paper seeks to add another variant to the microscopic counting – we ‘count’ twisted half-BPS states in theories with $\mathcal{N} = 4$ supersymmetry where the twist does not commute with the orbifolding group. This is as a prelude to considering situations where the twist does not commute with supersymmetry.

We consider four dimensional CHL \mathbb{Z}_n -orbifolds with $\mathcal{N} = 4$ supersymmetry [7, 8]. These models are asymmetric orbifolds [9, 10] constructed by starting with a heterotic string compactified on a $T^4 \times S^1 \times \tilde{S}^1$ and then quotienting the theory by a \mathbb{Z}_n transformation which involves a $1/n$ shift along the \tilde{S}^1 . The \mathbb{Z}_n symmetry has a non-trivial action on the internal conformal field theory coordinates describing the heterotic compactification on T^4 . A large class of such models were constructed in [11, 12] and were shown to be dual to a type II description compactified on $K3 \times S^1 \times \tilde{S}^1$ via string-string duality [13, 14].

By construction CHL models possess maximal supersymmetry and fewer massless vector multiplets at generic points in the moduli space. The requirement of maximal supersymmetry restricts one to consider symplectic automorphisms on $K3$. Symplectic automorphisms leave the holomorphic $(2, 0)$ forms invariant and hence preserve supersymmetry. The action of these symmetries have fixed points on the $K3$ surface and is accompanied by translations on the circle to avoid quotient singularities. So the allowed groups

must faithfully represent translations in \mathbb{R}^2 which implies that the quotienting group has to be abelian [15]. The possible abelian groups that act symplectically on $K3$ were classified and the action of the group on the $K3$ cohomology was calculated [16]. Once the action on the cohomology is determined one uses string-string duality to map the action to the Heterotic side. The map is allowed provided the supergravity side is free from fixed points, i.e the action on $K3$ must be accompanied by shifts on the torus.

The work of Mukai [17] opened up the possibility that non-abelian groups can act as symplectic automorphisms on the $K3$ surface. A couple of years ago Garbagnati [18] constructed elliptic $K3$ surfaces that admit dihedral group as symplectic automorphisms. These automorphisms are constructed by combining automorphisms which act both on the base and the fiber such that the resulting action is symplectic. In particular [18] determined the ranks of the invariant sublattice and the orthogonal complement and identified the orthogonal complement to the invariant sublattice with the lattices in [19]. However, for compactifications down to four dimensions one cannot quotient by a non-abelian group since these groups do not represent translations faithfully. However, one can consider the theory to be on special points in the moduli space that admit non-abelian symmetries and quotient by the commutator subgroup, which is abelian.

In this paper, we consider the CHL \mathbb{Z}_n -orbifold models ($3 \leq n \leq 6$ at special points in the moduli space where they admit dihedral $D_n = \mathbb{Z}_n \rtimes \mathbb{Z}_2$ symmetry³). The \mathbb{Z}_n subgroup is the commutator subgroup of the D_n group and may be quotiented. The special points in moduli space are specified by the elliptic $K3$ surfaces that admit $D_n, 3 \leq n \leq 6$ symmetries constructed in [18]. Since the action of \mathbb{Z}_n group is known on the $K3$ side, we map it to the heterotic string using the string-string duality. We then construct the CHL \mathbb{Z}_n orbifold in the heterotic picture and let the additional \mathbb{Z}_2 symmetry act as a twist in the partition function of the orbifolded theory. These twist symmetries are identical to the ones considered in [11, 12] but without shifts. For $\mathcal{N} = 4$ supersymmetry to be preserved these twists must commute with all the unbroken supersymmetries of the theory. Such twists have been considered in the $g \in \mathbb{Z}_n$ twisted partition function [20] for unorbifolded theories, which counts the index/degeneracy⁴ of elementary string states when the theory is restricted to special points in moduli space. The g -twisted helicity index is defined as

$$B_{2m}^g = \frac{1}{2m!} \text{Tr}[g(-1)^{2\ell}(2\ell)^{2m}] , \quad (1)$$

³In our notation, D_n is the dihedral group of order $2n$, see section 2.

⁴Both are identical for the cases considered in this paper.

where g generates a symmetry of finite order, ℓ is the third component of angular momentum of a state in the rest frame, and the trace is taken over all states carrying a given set of charges. States which break less than or equal to $4m$ g -invariant supersymmetries give non-vanishing contributions to B_{2m}^g [20]. For the case of 1/2 BPS states that we consider in this paper, the relevant index is B_4^g .

For our case, the choice of the moduli space that has dihedral symmetry is compatible with the $g \in \mathbb{Z}_2$ twist. The other requirement that the physical charges have to be g invariant is met by requiring the charges Q to take values from lattices invariant under Dihedral symmetry [18, 19]. This choice is also compatible with the orbifold action, since these lattices possess invariance under both \mathbb{Z}_2 and \mathbb{Z}_n actions. Thus one meets the requirements for the twist and orbifold action to be well defined.

We count the degeneracy of electrically charged 1/2 BPS elementary string states for a fixed charge Q in these theories following the method described in [21]. The \mathbb{Z}_2 twisted partition function in the \mathbb{Z}_n orbifold theories receives contribution only from the orbifold untwisted sector for odd n and additionally from the orbifold sector twisted by the element $h^{n/2}$ for even n . From the point of view of the dihedral group, for even n , the element $h^{n/2}$ is a nontrivial center of the group and commutes with every element. We derive a generating function for these degeneracies and find that it has the expected asymptotic limit. The generating function for these twisted 1/2 BPS also forms a Mathieu representation as it did for the abelian cases [5, 22].

The paper is organised as follows. Following the introductory section, in section 2, we give a pedagogical introduction to non-abelian orbifolds and define the twisted partition function to indicate the contributing orbifold twisted sectors. It is followed by the construction of CHL \mathbb{Z}_n orbifolds in the heterotic picture and the derivation of the half-BPS degeneracies of $g \in \mathbb{Z}_2$ twisted BPS states in section 3. In section 4, we provide a connection with the sporadic Mathieu group M_{24} . We conclude with a summary of our results in section 5.

2 Non-abelian orbifolds

In this section we describe the standard CFT approach for constructing non-abelian orbifold theories. For a general description of orbifolds in string theory see [23–29]. For some phenomenological model building approaches based on non-abelian orbifold string theories see [30, 31]. Orbifold CFT's are generally constructed by considering a theory \mathcal{T} which admits a finite discrete symmetry group G consistent with its allowed interactions and then

forming a quotient theory \mathcal{T}/G . When the orbifolding group is abelian the entire group acts as the symmetry group whereas in non-abelian orbifolds the symmetry group is $G/[G, G]$, where $[G, G]$ is the commutator subgroup of G . As an example, for Dihedral groups D_n of order $2n$, the quotienting group is the cyclic group \mathbb{Z}_n of order n . Before proceeding further, it is useful to define some notations. Let us denote the worldsheet coordinate as $X(\tau, \sigma)$, with τ and σ being the “space” and “time” directions of the torus. By

$${}_g \square_h \equiv \text{Tr}_{\mathcal{H}_h}(g q^H), \quad (2)$$

we mean the following closed string boundary conditions are applied simultaneously.

$$\begin{aligned} X(\tau + 2\pi, \sigma) &= g \cdot X(\tau, \sigma) \\ X(\tau, \sigma + 2\pi) &= h \cdot X(\tau, \sigma). \end{aligned} \quad (3)$$

$\text{Tr}_{\mathcal{H}_h}$ denotes the trace taken in a Hilbert space sector \mathcal{H}_h corresponding to a spatial twist element h . H denotes the Hamiltonian of the theory. We also denote $|G|$ as the order of the group G . The module ${}_g \square_h$ is not well defined for $gh \neq hg$ as we will explain below.

For the CFT to be well defined, the states of the theory must be invariant under the action of the group. Therefore one projects onto G -invariant states by defining a projection operator

$$P = \frac{1}{|G|} \sum_{g \in G} g. \quad (4)$$

The projection is implemented by including g in the trace and then by summing over all time twists. The inclusion of g in the trace amounts to twisting the fields by g along the time direction, i.e $g \cdot X(\tau, \sigma) = X(\tau + 2\pi, \sigma)$. The contribution to the partition function from the spatially untwisted sector of the orbifold CFT is then given by

$$Z_{\mathcal{H}_e} = \frac{1}{|G|} \sum_{g \in G}. \quad (5)$$

Modular invariance under $SL(2, \mathbb{Z})$ transformations requires the addition of spatially twisted sectors ${}_e \square_h$, i.e sectors where fields satisfy $h \cdot X(\tau, \sigma) = X(\tau, \sigma + 2\pi)$. Each of these spatially h -twisted sectors corresponds to a distinct Hilbert space \mathcal{H}_h and one must project onto the group invariant

states within every Hilbert space. This would mean that the fields would have simultaneous boundary conditions due to the action of g and h .

$$\begin{aligned} X(\tau, \sigma + 2\pi) &= hX(\tau, \sigma) & X(\tau + 2\pi, \sigma) &= gX(\tau, \sigma) \\ gX(\tau, \sigma + 2\pi) &= ghX(\tau, \sigma) & hX(\tau + 2\pi, \sigma) &= gX(\tau, \sigma) \\ gX(\tau, \sigma + 2\pi) &= ghg^{-1}gX(\tau, \sigma) & hX(\tau + 2\pi, \sigma) &= hgh^{-1}hX(\tau, \sigma) \\ X(\tau + 2\pi, \sigma + 2\pi) &= ghX(\tau, \sigma) & X(\tau + 2\pi, \sigma + 2\pi) &= hgX(\tau, \sigma). \end{aligned} \quad (6)$$

From the above equations, one can see that the action of g takes the string in the Hilbert space \mathcal{H}_h to the Hilbert space $\mathcal{H}_{ghg^{-1}}$. When g and h do not commute these Hilbert spaces are different. The elements h and $h' = ghg^{-1}$ are in the same conjugacy class and hence the projection operator would mix Hilbert spaces corresponding to elements that belong to a given conjugacy class. Thus, one is unable to do a full group invariant projection within the Hilbert spaces in the spatially twisted sectors. In the operator language, the presence of a time twist g that doesn't commute with the spatial twist element h would not allow simultaneous diagonalization of their respective matrix representations. Nevertheless one can choose a basis for g and it acts on the oscillators and eventually on the vacuum. As explained above, the vacuum is not left invariant and the vacuum in \mathcal{H}_h taken to the vacuum in $\mathcal{H}_{ghg^{-1}}$. So the trace would be over an off-diagonal matrix with diagonal entries zero and hence would vanish. Or equivalently, the path integral vanishes due to the inconsistent boundary condition (6). Since the spatially twisted sectors are not invariant under the full group. For a given spatially twisted sector \mathcal{H}_h one identifies the little group N_h consisting of elements that commute with h and project onto states invariant under the little group

$$Z_{\mathcal{H}_h} = \frac{1}{|N_h|} \sum_{g \in N_h} {}_g \square_h. \quad (7)$$

The various spatially twisted sectors in a given conjugacy class are treated in equal footing and hence the spatially twisted sectors are labelled by their conjugacy class C_i instead of the group element itself. This follows from “naive” modular invariance ⁵.

$$Z_{C_i} = \frac{1}{|C_i|} \sum_{h \in C_i} Z_{\mathcal{H}_h} = \frac{1}{|C_i|} \sum_{h \in C_i} \left(\frac{1}{|N_h|} \sum_{g \in N_h} {}_g \square_h \right) \quad (8)$$

The group invariant states in the theory are formed by taking a linear combination of states from a sector twisted by a group element g and all other

⁵modular invariance under $PSL(2, \mathbb{Z})$ transformations, It is naive because the modular transformation $\tau \rightarrow \tau + n$ can introduce anomalous phases that could spoil modular invariance

sectors conjugate to it. The full partition function is then given by summing over all the conjugacy classes

$$Z_{\mathcal{T}/G} = \sum_{C_i} Z_{C_i} \quad (9)$$

Since for any group G , the order of the little group N_h is the same for every element $h \in C_i$ ⁶, we have $|G| = |N_h||C_i|$ for every conjugacy class C_i . Thus the full CFT partition function for a general non-abelian orbifold theory can also be written as

$$Z_{\mathcal{T}/G} \equiv \frac{1}{|G|} \sum_{\substack{g,h \in G \\ gh=hg}} {}_g \square_h. \quad (10)$$

We summarize some properties of Dihedral groups which will be useful later. The dihedral group denoted as D_n is of order $2n$. One has the presentation,

$$D_n \cong \langle h, g | h^n = e, g^2 = e, ghg = h^{-1} \rangle. \quad (11)$$

where h, g generate of \mathbb{Z}_n and \mathbb{Z}_2 symmetries respectively. The group elements are given by $D_n = \{e, h, h^2, \dots, h^{n-1}, g, gh, gh^2, \dots, gh^{n-1}\}$. The \mathbb{Z}_2 generator acts as an inversion on the axes of reflection, all the elements of the form gh^j are of order 2, i.e. $(gh^j)^2 = 1$. The properties of dihedral group depend on whether n is even or odd. For odd n , D_n has $\lfloor n/2 \rfloor + 2$ conjugacy classes are given by (the little groups N_{c_i} for each element c_i in C_i are indicated beside)

$$\begin{aligned} C_0 &= \{e\}; & N_e &= D_n \\ C_1 &= \{g, gh, gh^2, \dots, gh^{n-1}\}; & N_{c_1} &= \{e, c_1\} \\ C_k &= \{h, h^{n-1}\}, \{h^2, h^{n-2}\}, \dots, \{h^{\lfloor n/2 \rfloor}, h^{\lfloor n/2 \rfloor+1}\}; & N_{c_k} &= \mathbb{Z}_n \end{aligned} \quad (12)$$

For even n , D_n has $n/2 + 3$ conjugacy classes which are given by

$$\begin{aligned} C_0 &= \{e\}; & N_e &= D_n \\ C_1 &= \{h^{n/2}\}; & N_{c_1} &= D_n \\ C_2 &= \{g, gh^2, gh^4, \dots, gh^{n-2}\}; & N_{c_2} &= \{e, c_2, h^{n/2}, c_2 h^{n/2}\} \\ C_3 &= \{gh, gh^3, gh^5, \dots, gh^{n-1}\}; & N_{c_3} &= \{e, c_3, h^{n/2}, c_3 h^{n/2}\} \\ C_k &= \{h, h^{n-1}\}, \{h^2, h^{n-2}\}, \dots, \{h^{\lfloor n/2 \rfloor-1}, h^{\lfloor n/2 \rfloor+1}\}; & N_{c_k} &= \mathbb{Z}_n \end{aligned} \quad (13)$$

⁶this is because every element in a conjugacy class has the same order, a group element h is of order n if $h^n = 1$

The group invariant projection operator for D_n has the property

$$\begin{aligned} P_{D_n} &= \frac{1}{2n} \left(\sum_{j=0}^{n-1} h^j + \sum_{j=0}^{n-1} gh^j \right) \\ &= \frac{1}{2} \sum_{k=0}^1 g^k \left(\frac{1}{n} \sum_{j=0}^{n-1} h^j \right) \\ &= P_{\mathbb{Z}_2} \cdot P_{\mathbb{Z}_n}, \end{aligned} \tag{14}$$

which follows from the property of the group elements (11). Even though the element g does not commute with elements $h \in \mathbb{Z}_n$, it commutes with the projector of \mathbb{Z}_n . Thus if we take g to be a twist, it *commutes with the orbifold projection*. The \mathbb{Z}_n partition function is given by

$$Z_{T/\mathbb{Z}_n} = \frac{1}{n} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} h^j \square_{h^k} \tag{15}$$

Twisting the partition function by $g \in \mathbb{Z}_2$ amounts to insertion of g in the trace,

$$\text{Tr}_{\mathcal{H}_h}(g q^H) \tag{16}$$

By the arguments given in (6) only the following terms contribute to the trace,

$$Z_{T/\mathbb{Z}_n}^g = \frac{1}{n} \left[\sum_{j=0}^{n-1} gh^j \square_e + \delta_{\frac{n}{2}, [\frac{n}{2}]} \sum_{j=0}^{n-1} gh^j \square_{h^{n/2}} \right] \tag{17}$$

The second sets of terms are there only for even n as can be seen from (13). We refer to this partition function as the “twisted” partition function. Since the twist generating group \mathbb{Z}_2 does not commute with the orbifold group \mathbb{Z}_n , we refer to it as a non-commuting twist. In the following sections, we discuss the orbifold action and then evaluate (17) for the CHL \mathbb{Z}_n -orbifolds.

3 Computing the Twisted Partition Function

We adapt the half-BPS counting method of Sen [21] to compute the twisted partition function. In the notation $D_n = \mathbb{Z}_n \rtimes \mathbb{Z}_2 = H \rtimes G$, H is the commutator subgroup of D_n which is also the orbifolding group. G represents an additional symmetry of the theory that appears at special points in the moduli spaces. The CHL \mathbb{Z}_n -orbifold can be described as an asymmetric orbifold of the heterotic string compactified on $T^4 \times T^2$. The \mathbb{Z}_n symmetry acts as a shift on one of the circles in the T^2 and as a symmetry transformation

on the rest of the CFT involving the T^4 coordinates and the 16 left-moving world-sheet scalars associated with the $E_8 \times E_8$ gauge group. The action of a group element h of the orbifold group H is the combination of a shift a_h and a rotation R_h acting on the Narain Lattice $\Gamma^{(22,6)}$. The action of the twist $g \in \mathbb{Z}_2$ on the $K3$ side is known [11, 16] and has been used to compute twisted indices in [20]. g leaves 14 of the 22 2-cycles of $K3$ invariant, in other words it exchanges the two E_8 's. Furthermore g is not accompanied by shifts. The $g \in \mathbb{Z}_2$ insertion in trace requires the physical charges Q to be g -invariant and the orbifolding requires it to be compatible with the \mathbb{Z}_n orbifold projection. Hence, we let Q takes values in the lattices that are invariant under $D_n = \mathbb{Z}_n \rtimes \mathbb{Z}_2$ symmetry [19]. For the rest of the computation we fix the value of Q . Once this is done the twist g has no further action on the lattice.

The set of $R_h \forall h \in H$ forms a group that describes the rotational part of H and is represented as R_H . To preserve $\mathcal{N} = 4$ supersymmetry both R_H and g must act trivially on the right movers. In the $K3$ side this is enforced by requiring the respective automorphisms to be symplectic. The group H leaves $22 - k$ of the 22 left moving directions invariant, where k is the number of directions that are not invariant under H . Then, R_H can be characterized by $k/2$ phases $\phi_j(h)$ with $j = 1, 2, \dots, k/2$. The complex coordinates X^j represent the planes of rotation and the effect of the rotation R_H is to multiply the complex oscillators by phases.

The groups also act on the Narain lattice $\Gamma^{(22,6)}$ and leave a sublattice Λ_\perp invariant. The orthogonal complement to Λ_\perp is denoted as Λ_\parallel . To preserve $\mathcal{N} = 4$ supersymmetry the right movers take their charge values only from the invariant part of the lattices and the non-invariant part of the lattice is only due to the k left moving directions that are not invariant under the action of the group.. Thus $\text{rank}(\Lambda_{\perp L}) = 22 - k$, $\text{rank}(\Lambda_\parallel) = k$ and $\text{rank}(\Lambda_{\perp R}) = 6$.⁷ The total number of $U(1)$ gauge fields in the theory is given by $\text{rank}(\Lambda_\perp) = 22 + 6 - k$. For the \mathbb{Z}_n groups, the values of k can be read off from Table 1.

We recollect some lattice definitions from [21] for convenience. Let V be the $22 + 6$ dimensional vector space in which the Narain lattice $\Gamma^{(22,6)}$ is embedded. The action of a given group element $h \in \mathbb{Z}_n$ on V leaves a subspace $V_\perp(h)$ invariant. The planes of rotation lie along a subspace denoted as $V_\parallel(h)$. It is clear that $V_\parallel(h)$ and $V_\perp(h)$ are mutually orthogonal to each other. The action of the entire group thus separates the vector space V into an invariant subspace V_\perp and its orthogonal complement V_\parallel which are

⁷This corresponds to the six graviphotons that arise from the toroidal compactification.

G	rank(Λ_{\parallel})	rank($\Lambda_{\perp L}$)
\mathbb{Z}_2	8	14
\mathbb{Z}_3	12	10
\mathbb{Z}_4	14	8
\mathbb{Z}_5	16	6
\mathbb{Z}_6	16	6
\mathbb{Z}_7	18	4
\mathbb{Z}_8	18	4
$D_2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$	12	10
$\mathbb{Z}_2 \times \mathbb{Z}_4$	16	6
$\mathbb{Z}_2 \times \mathbb{Z}_6$	18	4

Table 1: For the abelian groups the ranks of the invariant sublattice and the orthogonal complement are given in [32].

defined as⁸

$$V_{\perp} = \bigcap_{h \in \mathbb{Z}_n} V_{\perp}(h) \quad ; \quad V_{\parallel} = \bigcup_{h \in \mathbb{Z}_n} V_{\parallel}(h) \quad (18)$$

The invariant sublattice Λ_{\perp} and its orthogonal complement Λ_{\parallel} are defined as

$$\Lambda^{\mathbb{Z}_n} := \Lambda_{\perp} = \Gamma \bigcap V_{\perp} \quad ; \quad \Lambda_{\mathbb{Z}_n} := \Lambda_{\parallel} = \Gamma \bigcap V_{\parallel}. \quad (19)$$

and

$$\Lambda_{\perp}(h) = \Gamma \bigcap V_{\perp}(h) \quad ; \quad \Lambda_{\parallel}(h) = \Gamma \bigcap V_{\parallel}(h), \quad (20)$$

where $\Lambda_{\perp}(h)$ is the lattice component left invariant by a group element h and $\Lambda_{\parallel}(h)$ is the orthogonal complement. The ranks of these lattices are the dimensions of their respective vector spaces.

In the following we describe the heterotic construction of the counting [21] in the untwisted sector as the non-commuting twist obtains no contribution from the twisted sectors. The projection is unto states invariant under the orbifold group \mathbb{Z}_n . For individual elements, $h \in \mathbb{Z}_n$ there will be a non-trivial shift vector along with the rotation. In order to obtain expressions for $g \in \mathbb{Z}_2$ one has to just put the shift vectors a_g to zero. For composite elements like gh one has a rotation due to h followed by a reflection on the axes of rotation

⁸The sublattice that is invariant under a group G acting on a lattice, Λ , is usually denoted by Λ^G and its orthogonal complement by Λ_G .

by g and there is also a shift on the lattice due to h , this follows from the group multiplication law. However one does not need such explicit details in the computation as we will show later.

As is known, the momenta and windings in the compact directions of the theory takes values in the Narain lattice $\Gamma^{(22,6)}$. The (left,right) components of the momentum vector are denoted as $\vec{P} = (\vec{P}_L, \vec{P}_R)$. Let N_L, N_R be the total level of left moving and right moving oscillator excitations respectively. For a BPS state, the right movers are kept at the lowest eigenvalue allowed by GSO projection, i.e $N_R = 0$. The level matching condition in the untwisted sector is

$$N_L - 1 + \frac{1}{2}(\vec{P}_L^2 - \vec{P}_R^2) = 0. \quad (21)$$

Let $Q = (\vec{Q}_L, \vec{Q}_R)$ denote the projection of \vec{P} along V_\perp and $P_{\parallel} = (\vec{P}_{\parallel L}, 0)$ the projection of \vec{P} along V_{\parallel} . In an orbifold theory such as this one, only the components of P along V_\perp can act as sources for electric fields. Since $\mathcal{N} = 4$ supersymmetry requires the right-moving momenta to take values only from the invariant sublattice, \vec{P}_R lies entirely along V_\perp , we deduce $\vec{P}_R = \vec{Q}_R$. It is then clear that \vec{P}_L has the projection \vec{Q}_L along V_\perp and $\vec{P}_{\parallel L}$ along V_{\parallel} . Thus \vec{P}_L has a orthogonal decomposition

$$\vec{P}_L = \vec{Q}_L + \vec{P}_{\parallel L}. \quad (22)$$

Writing $N = \frac{1}{2}(\vec{Q}_R^2 - \vec{Q}_L^2)$ the level matching condition in the untwisted sector (21) reads

$$N_L - 1 + \frac{1}{2}\vec{P}_{\parallel L}^2 = N. \quad (23)$$

Note that, the information that the charge vector should take values on some specific lattice has gone into N , and the orbifold projection proceeds in the usual way. The counting of the number of \mathbb{Z}_n -invariant BPS states for a given charge Q is then done by implementing the group invariant projection. The contribution to the trace with a group element $h \in \mathbb{Z}_n$ inserted comes only from those $\vec{P}_{\parallel L}$ which are invariant under the action of h , i.e from those $\vec{P}_{\parallel L}$ which satisfy the condition

$$\vec{P}_{\parallel L} \in V_\perp(h). \quad (24)$$

Furthermore, two vectors P and P' in Λ which may correspond to the same charge vector Q would differ by a constant vector. Hence the allowed values of $\vec{P}_{\parallel L}$ for a given charge vector \vec{Q} are of the form

$$\vec{P}_{\parallel L} = \vec{K}(Q) + \vec{p}, \quad (25)$$

where $\vec{p} \in \Lambda_{\parallel}$ and $\vec{K}(Q) \in (\Lambda_{\parallel}^*/\Lambda_{\parallel})$ is a constant vector that lies in the unit cell of Λ_{\parallel} . The total momentum vector can thus be decomposed as

$$\vec{P} = \vec{P}_L + \vec{P}_R = (\vec{Q}_L + \vec{P}_{\parallel L}) + \vec{Q}_R = \vec{Q} + (\vec{p} + \vec{K}(Q)). \quad (26)$$

When a group element h acts on the vacuum carrying such a momentum \vec{P} it will produce a phase [9]

$$h |P\rangle = e^{2\pi i \vec{a}_h \cdot \vec{Q}} e^{-2\pi i \vec{a}_{hL} \cdot (\vec{p} + \vec{K}(Q))} |P\rangle \quad (27)$$

where \vec{a}_h is the shift vector on the lattice associated with the group element h and \vec{a}_{hL} is its left moving component. Note that there no phases associated with g since $a_g = 0$. The negative sign is due to the signature of the lattice. Thus we can now express the degeneracy of BPS states in the untwisted sector of the orbifold carrying a charge $\vec{Q} \in \Gamma_{\perp}$ as

$$d(Q) = \frac{16}{|\mathbb{Z}_n|} \sum_{h \in \mathbb{Z}_n} \sum_{N_L=0}^{\infty} d^{osc}(N_L, h) e^{2\pi i \vec{a}_h \cdot \vec{Q}} \sum_{\substack{\vec{p} \in \Lambda_{\parallel} \\ \vec{p} + \vec{K}(Q) \in V_{\perp}(h)}} e^{-2\pi i \vec{a}_{hL} \cdot \vec{p}} \delta_{N_L - 1 + \frac{1}{2}(\vec{p} + \vec{K}(Q))^2, N} \quad (28)$$

where $d^{osc}(N_L, h)$ is the number of ways one can construct oscillator level N_L from the 24 left-movers weighted by the action of h . The factor of 16 accounts for the degeneracy of a single BPS multiplet. The \vec{Q} -dependent phase in the above equation prevents us from directly computing the generating function of the degeneracies. Sen [21] evaluates the degeneracy treating \vec{Q} and N as independent variables in the right hand side of the above equation and calling it $F(Q, \hat{N})$. Of course, setting $\hat{N} = N = \frac{1}{2}Q^2$ in $F(Q, \hat{N})$, one recovers $d(Q)$. The symbol \hat{N} is used to indicate that N is treated as an independent variable.

$F(Q, \hat{N})$ counts the number of states in the CFT which carry a given charge Q , with right-movers in the ground state. The CFT has $\bar{L}_0 - L_0$ eigenvalue $\hat{N} - \frac{1}{2}Q^2$ which takes integer values from one-loop modular invariance. The integer condition for level matching is satisfied only after summing over all the h in the trace. A partition function can be defined as follows:

$$\tilde{F}(Q, \mu) = \sum_{\hat{N}} F(Q, \hat{N}) e^{-\mu \hat{N}}, \quad (29)$$

where \hat{N} runs over values for which $F(Q, \hat{N})$ is non-zero.

$\tilde{F}(Q, \mu)$ acts as a generating function for the degeneracy of electrically charged 1/2 BPS states in the theory. Substituting for $F(Q, \hat{N})$ from equation (28) one obtains

$$\begin{aligned} \tilde{F}(Q, \mu) = \frac{16}{|\mathbb{Z}_n|} \sum_{\hat{N}} \left[\sum_{h \in \mathbb{Z}_n} \sum_{N_L=0}^{\infty} d^{osc}(N_L, h) e^{2\pi i \vec{a}_h \cdot \vec{Q}} e^{-2\pi i \vec{a}_{hL} \cdot \vec{K}(Q)} \right. \\ \left. \sum_{\substack{\vec{p} \in \Lambda_{||} \\ \vec{p} + \vec{K}(Q) \in V_{\perp}(h)}} e^{-2\pi i \vec{a}_{hL} \cdot \vec{p}} \delta_{N_L - 1 + \frac{1}{2}(\vec{p} + \vec{K}(Q))^2, \hat{N}} \right] e^{-\mu \hat{N}} . \end{aligned} \quad (30)$$

The sum over \hat{N} can be carried out and it gets rid of the Kronecker delta function to give

$$\tilde{F}(Q, \mu) = \frac{16}{|\mathbb{Z}_n|} \sum_{h \in \mathbb{Z}_n} e^{2\pi i \vec{a}_h \cdot \vec{Q}} e^{-2\pi i \vec{a}_{hL} \cdot \vec{K}(Q)} \tilde{F}^{osc}(h, \mu) \tilde{F}^{lat}(Q, h, \mu) . \quad (31)$$

where the oscillator and lattice contribution to the partition function as

$$\begin{aligned} \tilde{F}^{osc}(h, \mu) &= \sum_{N_L=0}^{\infty} d^{osc}(N_L, h) e^{-\mu(N_L - 1)} , \\ \tilde{F}^{lat}(Q, h, \mu) &= \sum_{\substack{\vec{p} \in \Lambda_{||} \\ \vec{p} + \vec{K}(Q) \in V_{\perp}(h)}} e^{-2\pi i \vec{a}_{hL} \cdot \vec{p}} e^{-\frac{1}{2}\mu(\vec{p} + \vec{K}(Q))^2} . \end{aligned} \quad (32)$$

Note that \tilde{F}^{osc} has no dependence on Q while \tilde{F}^{lat} depends weakly on \vec{Q} only through $\vec{K}(Q)$.

The inverse of the partition function gives the degeneracy

$$F(Q, \tilde{N}) = \frac{1}{2\pi i} \int_{\epsilon-i\pi}^{\epsilon+i\pi} d\mu \tilde{F}(Q, \mu) e^{\mu \tilde{N}} \quad (33)$$

where $\mu = 2\pi\tau/i$ and ϵ is a real positive number. It has been argued in [21] that this integral receives its dominant contribution from a small region around the origin. Hence, we will take the $\mu \rightarrow 0$ limit later. The oscillator contribution is calculated easily by noting that the upon the action of a group element h the oscillator acquires a phase $e^{2\pi\phi_j(h)}$ ⁹

$$\tilde{F}^{osc}(h, \mu) = q^{-1} \left(\prod_{n=1}^{\infty} \frac{1}{1 - q^n} \right)^{24-k} \prod_{j=1}^{k/2} \left(\prod_{n=1}^{\infty} \frac{1}{1 - e^{2\pi i \phi_j(h)} q^n} \frac{1}{1 - e^{-2\pi i \phi_j(h)} q^n} \right), \quad (34)$$

⁹Note that the elements $h \in \mathbb{Z}_n$ are of cyclic type, i.e $h^n = 1$ for some $n \in \mathbb{Z}$, so the phases are all of type $\frac{p}{n}$ for some $p \in \mathbb{Z}$

where k is the number of non-invariant directions under \mathbb{Z}_n . When g is inserted into the trace, it will act on the oscillators. The phase and number of directions of rotation due to the elements in $\tilde{F}^{osc}(h, \mu)$ depends only on the order of the group element. In evaluating the oscillator contribution for

$$g \begin{array}{c} \square \\ e \end{array} + gh \begin{array}{c} \square \\ e \end{array} + gh^2 \begin{array}{c} \square \\ e \end{array} + \dots + gh^{n-1} \begin{array}{c} \square \\ e \end{array} \quad (35)$$

One notices that all the elements g, gh, \dots, gh^{n-1} are of order 2. Hence all of their oscillator contributions are identical to $g \begin{array}{c} \square \\ e \end{array}$. Since g exchanges the E_8 co-ordinates, the number of directions that are rotated (1) $k = 8$ and non zero phases $\phi_j(g) = 1/2$. Upon simplification, the oscillator contribution becomes

$$\tilde{F}^{osc}(g, \mu) = \frac{1}{\eta(\tau)^8 \eta(2\tau)^8}, \quad (36)$$

where

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \quad \text{with } q = e^{2\pi i \tau} = e^{-\mu}. \quad (37)$$

To write down the generating function, we need the lattice contribution due a particular group element h which is given by,

$$\tilde{F}^{lat}(Q, h, \mu) = \sum_{\substack{\vec{p} \in \Lambda_{||} \\ \vec{p} + \vec{K}(Q) \in V_{\perp}(h)}} e^{-2\pi i a_h \vec{a}_L \cdot \vec{p}} e^{-\frac{1}{2}\mu(\vec{p} + \vec{K}(Q))^2}. \quad (38)$$

We have already restricted the charges to take values on the D_n invariant lattices, hence g insertion has no further action on the lattice. When h is identity the conditions on $\vec{P}_{||L} = \vec{P} + \vec{K}(Q) \in V_{\perp}(h)$ is trivially satisfied since $V_{\perp}(e) = V$. For any other h , since we have $\dim V_{\perp}(h) < \dim(V)$, it follows that

$$\tilde{F}^{lat}(Q, h, \mu) \leq \tilde{F}^{lat}(\dim Q, e, \mu) \quad (39)$$

Therefore the dominant contribution is when $h = e$

$$\tilde{F}^{lat}(Q, e, \mu) = \sum_{\vec{p} \in \Lambda_{||}} e^{-\frac{1}{2}\mu(\vec{p} + \vec{K}(Q))^2}. \quad (40)$$

where the phase has disappeared as the identity element doesn't shift the vectors. As mentioned earlier, this lattice theta function depends on Q , only through the $\vec{K}(Q) \in \Lambda_{||}^*/\Lambda = \Lambda_{\perp}^*/\Lambda_{\perp}$ – thus there are only a finite number of lattice sums to consider.

Thus, combining the oscillator and lattice contributions (36) and (40) we get the result

$$\tilde{F}(Q, \mu) \sim \frac{16}{|\mathbb{Z}_n|} \frac{\tilde{F}^{lat}(Q, e, \mu)}{\eta(\tau)^8 \eta(2\tau)^8}, \quad (41)$$

with $\tau = i\mu/2\pi$. The nice thing about the right hand side of the above equation is that it depends only on $\vec{K}(Q)$. Thus, up to exponentially smaller terms corresponding to $h \neq e$, the right hand side is the generating function of g -twisted half-BPS states in the charge sector $\vec{K}(Q)$. This is the main result of this section.

This $g \in \mathbb{Z}_2$ twisted partition function counts g -twisted half-BPS states in a \mathbb{Z}_n orbifold theory, so naturally we expect these modular forms to have weights smaller than the ones obtained for the untwisted orbifold theories. We will check that this is indeed the case by taking the asymptotic limit of (41). The $\mu \rightarrow 0$ limit of Dedekind eta function

$$\eta(\mu) \simeq e^{-\frac{\pi^2}{6\mu}} \sqrt{\frac{2\pi}{\mu}} \quad (42)$$

and the lattice contribution (40) after doing a Poisson resummation is

$$\tilde{F}^{lat}(e, \mu) \simeq \frac{1}{\text{vol}_{\Lambda_{||}}} \left(\frac{\mu}{2\pi} \right)^{-\frac{k_{\mathbb{Z}_n}}{2}}, \quad (43)$$

up to exponentially suppressed terms. Thus (41) has $\mu \rightarrow 0$ limit

$$\lim_{\mu \rightarrow 0} \tilde{F}(\mu) \simeq \frac{16}{|\mathbb{Z}_n|} \frac{1}{\text{vol}_{\Lambda_{||}}} e^{2\pi^2/\mu} \left(\frac{\mu}{2\pi} \right)^{8 - \frac{k_{\mathbb{Z}_n}}{2}} \quad (44)$$

We compare the weights of the modular forms for the half-BPS states in \mathbb{Z}_n orbifolds [5, 21] and the modular forms for g twisted half-BPS states in \mathbb{Z}_n orbifolds

Group	$12 - \frac{k_{\mathbb{Z}_n}}{2}$	$8 - \frac{k_{\mathbb{Z}_n}}{2}$	$k_{\mathbb{Z}_n} = \text{rank}(\Lambda_{ })$
\mathbb{Z}_3	6	2	12
\mathbb{Z}_4	5	1	14
\mathbb{Z}_5	4	0	16
\mathbb{Z}_6	4	0	16

One can see from the above table that the weights for the g twisted half-BPS states are indeed smaller.

The other contribution for even n

For the even n , as noted in the end of section (2), we will get additional contribution from the orbifold twisted sector due to the element $h^{n/2}$.

$$g \begin{array}{|c|} \hline \square \\ \hline h^{n/2} \end{array} + gh \begin{array}{|c|} \hline \square \\ \hline h^{n/2} \end{array} + gh^2 \begin{array}{|c|} \hline \square \\ \hline h^{n/2} \end{array} + \dots + gh^{n-1} \begin{array}{|c|} \hline \square \\ \hline h^{n/2} \end{array}. \quad (45)$$

Here again, the oscillator contribution from each module is identical since the elements have the same order. The \mathbb{Z}_n groups, for n even have \mathbb{Z}_2 as a subgroup which would commute with the g twist in the partition function to give a $\mathbb{Z}_2 \times \mathbb{Z}_2$. This case was already computed in [22] (see Appendix A) in the context of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the result is

$$\tilde{F}^{osc}(\mu) = \frac{1}{\eta(2\tau)^{12}}. \quad (46)$$

We need to compute the lattice contribution in this $h^{n/2}$ sector. Here $\vec{P} \in \Lambda^{\mathbb{Z}_2}$ the lattice invariant under the \mathbb{Z}_2 generated by $h^{n/2}$ unlike the untwisted sector where it was in Λ . The charge vectors Q take value in the projection of \vec{P} along V_\perp . Thus, we have the lattice contribution given by

$$\tilde{F}^{lat}(Q, h, \mu) = \sum_{\substack{\vec{p} \in \Lambda_{||}^{\mathbb{Z}_2} \\ \vec{p} + \vec{K}(Q) \in V_\perp(h)}} e^{-2\pi i \vec{a}_{hL} \cdot \vec{p}} e^{-\frac{1}{2}\mu(\vec{p} + \vec{K}(Q))^2}, \quad (47)$$

where $\Lambda_{||}^{\mathbb{Z}_2} = \Lambda^{\mathbb{Z}_2} \cap V_{||}$ and $\vec{K}(Q) \in \Lambda_{||}^{\mathbb{Z}_2*}/\Lambda_{||}^{\mathbb{Z}_2}$. Again, the dominant contribution to the lattice sum occurs when $h = e$. The weight of the relevant modular form is now $6 - [k/2]$ where k is the rank of the lattice $\Lambda_{||}^{\mathbb{Z}_2}$. We estimate k using the relevant cycle shapes for the \mathbb{Z}_4 and \mathbb{Z}_6 orbifolds to be 6 and 8 respectively. when $n = 4$, the cycle shape for the element h is $1^4 2^2 4^4$. The invariant lattice has dimension $12 = 4 + 2 + 4$ and thus $\dim V_{||} = 24 - 12 = 12$. Elements that belong to $\Lambda_{||}^{\mathbb{Z}_2}$ are those that correspond to an h -eigenvalue equal to -1 . There are precisely six of them, two coming from the two-cycles and four from the four cycles. A similar analysis for the cycle shape $1^2 2^2 3^2 6^2$ for $n = 6$ shows that each three- and six-cycle contribute 2 elements with h^3 -eigenvalue equal to unity but h -eigenvalue not equal to unity and hence $k = 8$. A simple asymptotic counting as we did earlier then shows that this contribution is *larger* than the contribution from the untwisted sector given in Eq. (41).

4 Towards Mathieu representations

We just derived, in the previous section, a formula for the generating function of a family of non-commuting twists associated with the dihedral symmetry. Eq. (41) gives the final result of our computation. For the case of commuting twists, the counting of $\frac{1}{2}$ -BPS states saw the appearance of the sporadic Mathieu group, M_{24} [5, 22]. It is interesting to ask if the final result in Eq. (41) is associated with the Mathieu group. We shall show that this appears to be the case. Recall that the answer, up to overall numerical factors, was the ratio of a lattice theta function and an eta product. The connection with this eta product with the Mathieu group M_{24} was already shown in [5] and thus we need to show that the lattice sum has also a relation to M_{24} . Thus, the lattice contribution is of the form

$$\Theta_{\mathbb{Z}_n}^{\parallel}(\vec{K}(Q), \tau) \equiv \sum_{\vec{p} \in \Lambda_{\mathbb{Z}_n}} e^{-\frac{1}{2}\mu(\vec{p} + \vec{K}(Q))^2}, \quad (48)$$

where $\Lambda_{\mathbb{Z}_n}$ is the orthogonal complement to the \mathbb{Z}_n invariant sub lattice of the Lorentzian lattice $\Gamma^{(22,6)}$.

From the work of Nikulin and Mukai [16, 17], the lattice $\Lambda_{\mathbb{Z}_n}$ has the following properties:

- (i) It is a positive definite lattice¹⁰ with no roots and rank ≤ 19 [16].
- (ii) It can be primitively embedded into a Niemeier lattice (with at least one root) and the action of \mathbb{Z}_n can be extended to the full Niemeier lattice [33]. This is not quite appropriate for our considerations.
- (iii) However, in the appendix to Kondo's paper [33], Mukai refers to unpublished work of his that shows that $\Lambda_{\mathbb{Z}_n}$ can also be primitively embedded into the Leech lattice and the action of \mathbb{Z}_n can be extended to the full Leech lattice which we denote by L . One thus has the isomorphism: $\Lambda_{\mathbb{Z}_n} = L_{\mathbb{Z}_n}$, where $L_{\mathbb{Z}_n}$ is the orthogonal complement to the \mathbb{Z}_n invariant sub lattice of the Leech lattice, $L^{\mathbb{Z}_n}$.

This implies that the lattice sum can be mapped to a sum associated with a sub lattice of the Leech lattice. It is known that the automorphism group of the Leech lattice is the Conway group, Co_0 . Let L be the Leech lattice and L^G the sub lattice that is invariant under the action of a finite group $G \subset Co_0$ and L_G its orthogonal complement. While L is self-dual,

¹⁰According to the sign conventions of Nikulin and Mukai, it should be a negative definite lattice – this translates to a positive definite lattice in our convention. Thus the lattice sum is convergent.

neither L^G nor L_G are self-dual. Further, it is not true that $L = L^G \oplus L_G$. Following the discussion in the appendix of [9], we see that a generic lattice vector of L can be written as sum of two lattice vectors, one in L^{G*} and the other in L_G^* . Further, for every vector in L^{G*} , we can associate a vector in L_G^* modulo L_G . Thus, one has an isomorphism L^{G*}/L^G and L_G^*/L_G . Putting all this together, we see that a vector in L can be identified with two vectors in L^G and L_G along with an element of L^{G*}/L^G . This leads to the following decomposition of the theta function associated with the Leech lattice.

$$\Theta_L(\tau) = \sum_{a \in L^{G*}/L^G} \Theta_{L^G}^a(\tau) \Theta_{L_G}^a(\tau), \quad (49)$$

Setting $G = \mathbb{Z}_n \subset M_{23} \subset M_{24} \subset Co_0$, we see that $\Theta_{L_G}^a(\tau)$ is the lattice contribution that we computed for the non-commuting twist with the label a being identified with $\vec{K}(Q)$.

The lattice sums for elements with $\vec{K}(Q) = 0$ appear in the Monstrous moonshine correspondence of Conway and Norton [34]. The lattice sums associated with M_{24} have been explicitly worked out by Kondo and Tasaka [35]. The j -function associated with a conjugacy class, ρ , of M_{24} is of the form:

$$j_\rho(\tau) = \frac{\Theta_\rho(\tau)}{g_\rho(\tau)}, \quad (50)$$

where $g_\rho(\tau)$ are multiplicative eta products associated with the conjugacy class, ρ , of M_{24} [36, 37]. Kondo and Tasaka have given explicit formulae for the theta functions $\Theta_\rho(\tau)$ for all M_{24} conjugacy classes. These theta functions are lattice sums of sub lattices of the Leech lattice that are invariant under the cyclic group generated by the M_{24} group element in the conjugacy class ρ . Presumably, their methods can be extended to obtain explicit formulae for $\Theta_{\mathbb{Z}_n}^{\parallel}(\vec{K}(Q), \tau)$ as well. We end with the comment that $\Theta_\rho(\tau)$ is a modular from of a congruent subgroup of the full modular group. Under the action of $S : \tau \rightarrow -1/\tau$, it generates the lattice sum associated with L^{G*} which shows that $\Theta_{L^G}^a(\tau)$ can be generated in this fashion.

5 Discussion

In this paper, we have computed generating functions for non-commuting \mathbb{Z}_2 twists for CHL \mathbb{Z}_n orbifolds ($3 \leq n \leq 6$). The generating functions turn out to be ratios of the theta functions for the \mathbb{Z}_n group and eta products associated with the \mathbb{Z}_2 group. We have argued that the theta functions are also associated with the Mathieu group by using an isomorphism that maps the lattices

that appear to the sub lattices of the Leech lattice. When $n = 4$ and 6 , we find additional contributions also arise. Our computations did make use of the properties of the dihedral group. It would be interesting to extend this method to other nonabelian groups as well. On another note, this computation may also be extended to $1/4$ BPS states. One can use the symplectic automorphisms that act on the elliptic $K3$ directly in the Type IIA theory [20]. It will also be useful to consider twists that break supersymmetry, which means we would have to consider non-symplectic automorphisms on $K3$. Such twists will provide a controlled way to count BPS states in $\mathcal{N} = 2$ string theories.

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